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A SPECIAL FUCHSIAN SYSTEM CONNECTING SOME HILBERT PROBLEMS

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0. INTRODUCTION

This survey draws an investigation line from a special Fuchsian system of differential equations to special diophantine equations. E. PICARD [26] began to study more intensively the following special system of differential equations:

$$(0.1) \qquad D_{ij}\,F(u,v) = 0 \text{ on } \mathbb{C}^2 \setminus \boxslash = \mathbb{P}^2 - \bigtriangleup ,$$

$$(i,j) = (1,1),(1,2),(2,2) ,$$

$$D_{11} = \frac{\partial^2}{\partial u^2} + [9(u-1)u(v-u)]^{-1}\{3(-5u^2+4uv+3u-2v)\frac{\partial}{\partial u}+3(v-1)v\frac{\partial}{\partial v}+(u-v)\},$$

$$D_{12} = \frac{\partial^2}{\partial u \partial v} + [3(u-v)]^{-1}\{\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\},$$

$$D_{22} = \frac{\partial^2}{\partial v^2} + [9(v-1)v(u-v)]^{-1}\{3(u-1)u\frac{\partial}{\partial u}+3(-5v^2+4uv+3v-2u)\frac{\partial}{\partial v}-(u-v)\}.$$

$$\boxslash := \{(u,v) \in \mathbb{C}^2;\ uv(u-1)(v-1)u-v) \neq 0\},$$

$\bigtriangleup$:= six lines in $\mathbb{P}^2$ through pairs of four points in general position.

PICARD conjectured that the solutions of (1) should have an arithmetic meaning comparible with the role of elliptic integrals for plane cubic diophantine equations.

Fortunately D. HILBERT observed carefully the work of PICARD. We want to use some of the celebrated 23 problems of HILBERT [10] as an intuitive guide for a deeper arithmetic study of the solutions of (1). Via actual work of PARSHIN and VOJTA we discover at the end an interesting connection with FERMAT's equations.

Notations

$\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\bar{\mathbb{Q}}$, $\mathbb{Z}$: complex, real, rational, algebraic numbers, integers;

$\mathbb{P}^N$: (complex) projective space of dimension N;

$\mathbb{P}^N(L)$: Points of $\mathbb{P}^N$ with coordinates in the field $L \subseteq \mathbb{C}$;

$V(L) = V \cap \mathbb{P}^N(L)$, V a subvariety of $\mathbb{P}^{N'}$;

$\mathbb{G}l_n^+(\mathbb{R})$: elements of $\mathbb{G}l_n(\mathbb{R})$ with positive determinant;

$\mathbb{P}\mathbb{G}l_n$: the projective group $\mathbb{G}l_n/\mathbb{G}l_1$;

$\mathbb{P}G$: the image of G in $\mathbb{P}\mathbb{G}l_n$, G a subgroup of $\mathbb{G}l_n$;

$U((2,1),A)$: the unitary group of a hermitean form of signature (2,1) with coefficients in the ring $A \subseteq \mathbb{C}$ closed under complex conjugation;

$\mathcal{O}_L$: ring of integers in the number field L;

$\Gamma_L = U((2,1),\mathcal{O}_L)$ the full PICARD modular group of the imaginary quadratic number field L;

$K = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\rho)$: the field of EISENSTEIN numbers, ρ a primitive third unit root;

$\mathbb{B}^2 = \{(z_1, z_2) \in \mathbb{C}^2;\ |z_1|^2 + |z_2|^2 < 1\}$ the standard complex 2-ball;

$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \ldots$, $q = \exp(2\pi i\tau)$ the elliptic modular function defined on

$\mathbb{H} = \{\tau \in \mathbb{C};\ \mathrm{Im}\,\tau > 0\}$ the POINCARE upper half plane;

$\mathbb{H}_g = \{Z \in \mathbb{G}l_g(\mathbb{C});\ Z = {}^tZ,\ \mathrm{Im}\,Z > 0\}$ the generalized SIEGEL upper half plane;

$Sp(2g,A)$: symplectic group acting on $\mathbb{H}_g$, $A = \mathbb{R}$ or $\mathbb{Z}$;

$\mathcal{A}_g = Sp(2g,\mathbb{Z})\backslash\mathbb{H}_g$: the (non-compact) moduli space of g-dimensional (principally) polarized abelian varieties.

Hint: underlying on pages 5–9
should be: fat

1. The Central Problem

A central role in HILBERT's program plays the following

12-th HILBERT PROBLEM (Extension of Kronecker's theorem to arbitrary algebraic domains of rationality): "... to find and discuss those functions playing for arbitrary algebraic number fields the same role as the exponential function for the field of rational numbers and the elliptic modular function for imaginary quadradratic fields".

We remember to the elliptic modular function $j: \mathbb{H} \longrightarrow \mathbb{P}^1$. On $\mathbb{H}$ acts $Sl_2^+(\mathbb{R})$, $\mathbb{P}^1$ is the compactification of $\mathbb{H}/Sl_2(\mathbb{Z}) \cong \mathbb{C}$ and j describes the quotient map. We call $\tau \in \mathbb{H}$ a s i n g u l a r m o d u l e , if it is an isolated fixed point of (a $g \in$) $Sl_2^+(\mathbb{Q})$.

1.1 Theorem. Let τ be a point of $\mathbb{H}(\bar{\mathbb{Q}})$. Then

(i) $j(\tau)$ is a transcendent number iff τ is not a singular module;

(ii) If τ is a singular module then $j(\tau)$ is a class field (abelian extension, ray class field) of the imaginary quadratic field $\mathbb{Q}(\tau)$; it is the HILBERT class field (maximal unrarified abelian extension of $\mathbb{Q}(\tau)$), if $\mathbb{Z}1 + \mathbb{Z}\tau$ is a fractional ideal of $\mathbb{Q}(\tau)$.

The transcendence proof is due to SIEGEL [34]. This is an extension of the 7-th HILBERT PROBLEM asking for the transcendence of $e(\alpha) = \exp(\pi i\alpha)$ for $\alpha \in \bar{\mathbb{Q}} \setminus \mathbb{Q}$, affirmatively solved by Gelfond [8] and Schneider [28]. For a proof of (ii) we refer to SHIMURA

[33]. It is a part of KRONECKER's JUGENDTRAUM claiming the explicit construction of all abelian extensions of imaginary quadratic number fields by means of special values transcendendent functions. The JUGENDTRAUM appears as KRONECKER's problem in the first part of the 12-th HILBERT PROBLEM. It has been solved by TAKAGI [35] in 1921. Nowadays its solution is known as Main Theorem of Complex Multiplication, see [21].

2. Ball Uniformization of Surfaces

Let X be an algebraic surface over $\mathbb{C}$, compact for a moment, with at most quotient or ball *cusp singularities*. For a precise definition and classification of these singularities we refer to [16]. In this paper a special class of algebraic cycles is defined consisting of weigted irreducible curves and points on X. The weights are natural numbers or ∞. Finitely weighted points are *quotient points*. Infinitely weighted curves or points are *cusp curves* or *cusp points*, respectively. For a precise definition/classification we refer to [16] again. We call these special cycles *orbital cycles* (on X). A pair $\underline{X} = (X,\underline{D})$, $\underline{D}$ an orbital cycle on X, is called an *orbital surface*.

2.1 Example. Let $X = \mathbb{P}^2$, Q_1, Q_2, Q_3, Q_4 four points on $\mathbb{P}^2$ in general position, $L_{ij} = L_{ji}$ the line through Q_i, Q_j. By abuse of notations we define an orbital cycle by $\underline{D} = \sum_{1 \le i < j \le 4} 3L_{ij} + \sum_{k=1}^{4} \infty \cdot Q_k$, where the weights appear as coefficients. The pair $(\mathbb{P}^2, \underline{D})$ is an orbital surface.

2.2 Remark. If X is a multi-blowing up of $\mathbb{P}^2$ and D is supported by lines, then HIRZEBRUCH called D a (linear) arrangement, see [3].

If we omit on X and D all infinitely weighted curves and points, then we get an o p e n o r b i t a l s u r f a c e $\underline{X}_f = (X_f, \underline{D}_f)$.

2.3 Main example. The group $U((2,1),\mathbb{C})$ acts on the 2-ball $\mathbb{B} \subset \mathbb{P}^2$ via projective transformations. Let Γ be an arithmetic subgroup of $U((2,1),\mathbb{C})$. Then the quotient $X_f = \mathbb{B}/\Gamma$ is a (generally open) algebraic surface. The irreducible curves of the branch locus of the covering $\mathbb{B} \longrightarrow \mathbb{B}/\Gamma$ and the images of Γ-elliptic points on $\mathbb{B}$ are endowed with the corresponding ramification indices as weights (for a point after blowing up its preimages). Denote the corresponding (open) orbital cycle by $\underline{D}_f$. Then $\underline{X}_f = (X_f, \underline{D}_f)$ is an open orbital surface. We add the infinitely weighted compactification points of the Baily-Borel compactification $\widehat{\mathbb{B}/\Gamma}$ to the "closure" $\underline{\hat{D}}_f$ of $\underline{D}_f$. The pairs $(X_f, \underline{D}_f)$ or $(\hat{X}_f, \underline{\hat{D}}_f)$ are called o p e n or c o m p a c t i f i e d o r b i t a l b a l l q u o t i e n t s u r f a c e s, respectively. Sometimes they are denoted by $\underline{\mathbb{B}/\Gamma}$ or $\underline{\widehat{\mathbb{B}/\Gamma}}$, respectively.

A fundamental question is: Which orbital surfaces are ball quotients ? A precise answer can be considered as a special but important partial answer to the 22-nd HILBERT PROBLEM entitled "Uniformization of analytic relations by means of automorphic functions".

In order to find a constructive answer we introduced in [16] successively some invariants called (l o c a l) and g l o b a l) o r b i t a l h e i g h t s (respectively):

$$h_e, h_\tau\colon \{\text{orbital curves}\} \longrightarrow \mathbb{Q},$$

$$H_e, H_\tau\colon \{\text{orbital surfaces}\} \longrightarrow \mathbb{Q}.$$

O r b i t a l c u r v e s are defined to be irreducible 1-dimensional components $\underline{C}$ of (embedded !) orbital cycles $\underline{D}$ (with all weight informations of $\underline{D}$ along C). We give an implicit definition: If everything is smooth, compact and has trivial weight 1, then

$$h_e(\underline{C}) = h_e(C) = -(\text{Euler number of } C),$$

$$h_\tau(\underline{C}) = h_\tau(C) = -(\text{selfintersection index of } C \subset X),$$

$$H_e(\underline{X}) = H_e(X) = \text{Euler number of } X,$$

$$H_\tau(\underline{X}) = H_\tau(X) = \text{signature of } X.$$

If $\underline{f}\colon \underline{X} \longrightarrow \underline{X}'$, $\underline{C} \longrightarrow \underline{C}'$ are finite coverings, then the following degree formulas hold:

$H(\underline{X}) = \deg(f)H(\underline{X}')$, $h(\underline{C}) = \deg(\underline{f}|\underline{C})h(\underline{C}')$, $H = H_e, H_\tau$, $h = h_e, h_\tau$.

Roughly spoken, a f i n i t e c o v e r i n g $\underline{f}$ is a usual finite covering $f\colon X \longrightarrow X'$ compatible with weights in the sense of Galois theory. The definition restricts to orbital curves. Proofs of existence via explicit definitions can be found in [13] and the related literature.

2.4 Theorem ([16]). If $\underline{X} = (X,\underline{D}) = \underline{\widehat{\mathbb{B}/\Gamma}}$ is an orbital ball quotient, then it holds that

(2.5) $H_e(\underline{X}) = 3H_\tau(\underline{X}) > 0$, $h_e(\underline{C}) = 2h_\tau(\underline{C}) > 0$, $\underline{C} < \underline{D}$.

The discrete subgroup $\mathbb{P}\Gamma$ of $PU((2,1),\mathbb{C})$ is uniquely determined up to conjugation by the orbital surface $\underline{X}$.

So we found effective necessary conditions for an orbital surface to be a ball quotient. R.KOBAYASHI [20] gave sufficient conditions in another language. Until now there is no proof of the equivalence of Kobayashi's and our conditions. The following considerations indicate that they cannot be far away from each other.

The relations (2.5) can be understood as a system DIOPH(X,D) of diophantine equations and inequalities, if we write it down explicitly with all weights as variables. The coefficients depend only on geometric data of the supporting surface X and the supporting cycle D. It turns out that

2.6 Theorem ([16]). For any (admissible) pair (X,D) the system DIOPH(X,D) has at most finitely many solutions. These solutions can be calculated in an effective manner.

So a ball uniformization of a surface X with given branch locus D can only happen, if we find a solution of DIOPH(X,D) and there are, up to isomorphy, at most finitely many possibilities.

2.7 Corollary. If we fix infinite weights at the four triple points of $\subset \mathbb{P}^2$ defined in (0.1), then DIOPH($\mathbb{P}^2$,) has exactly one solution. The corresponding orbital surface ($\mathbb{P}^2$,) coincides with that of Example 2.1.

one variable, "...to show that in any case there exists a linear differential equation of the FUCHSian class with given singular locus and prescribed monodromy group". The final solution of this HILBERT PROBLEM has been given by H. ROEHRL [27]. We change over and restrict ourselves to the second dimension.

3.1 Theorem (M.YOSHIDA [38]). Let $\underline{X} = (X,\underline{D})$ be an orbital surface uniformizable by the ball with quotient map $p: \mathbb{B} \longrightarrow X$. Then the inverse p^{-1} of p is a (multivalued) developing map of a FUCHSian system of linear partial differential equations.

It means that there is locally a fundamental system of solutions I_0, I_1, I_2 extending analytically to $X \setminus D$, such that the multivalued map

$$(I_0 : I_1 : I_2) : X \setminus D \longrightarrow \mathbb{B} \subset \mathbb{P}^2,$$

$P \longmapsto (I_0(P) : I_1(P) : I_2(P))$, coincides with p^{-1} on $X \setminus D$. The FUCHSian system is called the u n i f o r m i z i n g e q u a t i o n of the orbital surface and the uniformizing group Γ is the m o n o d r o m y g r o u p of the system. Via solutions one gets a unitary representation of the fundamental group

$$\pi_1(X \setminus D) \longrightarrow \mathbb{P}\Gamma \subset \mathbb{P}\mathrm{Gl}_3(\mathbb{C}).$$

YOSHIDA [38] found an effective method in order to determine a corresponding Fuchsian system. Together with 2.8 one gets

3.2 Theorem. The system (0.1) is a uniformizing equation of the orbital surface $(\mathbb{P}^2, \ )$. Its monodromy group is the PICARD modular group $\Gamma(\sqrt{-3})$.

Now one can check KOBAYASHI's conditions to see that the ball uniformization of $(\mathbb{P}^2, \underline{\quad})$ really exists. In [11] we proved more:

2.8 Proposition. Let $\Gamma(\sqrt{-3}) \subset U((2,1),\mathcal{O}_K)$, $K = \mathbb{Q}(\sqrt{-3})$, be the congruence subgroup corresponding to the ideal $(1-\varrho)$, $\varrho = e^{2\pi i/3}$, of $\mathcal{O}_K$. Then the orbital surfaces $\underline{\mathbb{B}/\Gamma(\sqrt{-3})}$ and $(\mathbb{P}^2, \underline{\quad})$ coincide.

For a proof it is convenient to use the following

2.9 Theorem ([16]). For the c_2-volume $c_2(\Gamma)$ of a fundamental domain of a ball lattice Γ it holds that

(2.10) $$c_2(\Gamma) = H_e(\underline{\mathbb{B}/\Gamma}), \quad c_2(\Gamma)/3 = H_\tau(\underline{\mathbb{B}/\Gamma}).$$

In [15] we presented an effective formula for $c_2(\Gamma_M)$, M an imaginary quadratic number field, in terms of a special value of L-series using arithmetic-geometric methods in the proof. Then one gets $H_e(\underline{\mathbb{B}/\Gamma(\sqrt{-3})})$, $H_\tau(\underline{\mathbb{B}/\Gamma(\sqrt{-3})})$, the Chern numbers $c_2(\widehat{\mathbb{B}/\Gamma(\sqrt{-3})}) = 3$, $c_1^2(\widehat{\mathbb{B}/\Gamma(\sqrt{-3})}) = 9$ after classification of elliptic points and cusps, and finally $\widehat{\mathbb{B}/\Gamma(\sqrt{-3})} = \mathbb{P}^2$ by the theory of surface classification.

3. Ball Uniformization and Differential Equations

In [38] M.YOSHIDA succeeded to solve a higher-dimensional version of the RIEMANN-HILBERT problem. The background is HILBERT's 21-st PROBLEM "Proof of the existence of linear differential equations with prescribed monodromy groups" set up for functions of

4. GAUSS-MANIN Connection

YOSHIDA's general approach lifting the Gauss-Schwarz of FUCHSian equations to higher dimensions has a classical origin in the work of PICARD and APPELL. Especially for linear arrangements of low-dimensional projective spaces a more immediate result known as PTDM-Theorem (due to PICARD, TERADA, MOSTOW, DELIGNE) would be sufficient for our purposes. We refer to [38], [3] and further literature given there. But we prefer to change over from the analytic viewpoint to an algebraic-geometric approach in order to find "algebraic solutions" of special FUCHSian equations represented by integrals on algebraic curves along cycles depending on on parameters u,v. The general framework of the corresponding algebraic theory is known as G a u s s - M a n i n c o n n e c t i o n of algebraic families of algebraic manifolds. For more details we refer to [11] in order to understand the rather explicit theory of algebraic families of curves involving differential equations.

Let $\mathcal{C}/T$ be a smooth algebraic family of smooth algebraic varieties all defined over the complex numbers, say. The relative DE RHAM complex is a sequence

$$\Omega^{\cdot}_{\mathcal{C}/T}: \quad \mathcal{O}_T \xrightarrow{d} \Omega^1_{\mathcal{C}/T} \xrightarrow{d} \Omega^2_{\mathcal{C}/T} \xrightarrow{d} \ \ldots .$$

Using open (affine, say) coverings one defines the ČECH complexes

$$C^{\cdot}(\Omega^q_{\mathcal{C}/T}): \quad C^0(\Omega^q_{\mathcal{C}/T}) \xrightarrow{\delta} C^1(\Omega^q_{\mathcal{C}/T}) \xrightarrow{\delta} C^2(\Omega^q_{\mathcal{C}/T}) \ \ldots$$

in the usual manner. Taking the limit along refinements of open coverings one gets the ČECH - DE RHAM bicomplex $C^{\cdot\cdot}(\Omega_{\mathcal{C}/T})$. The

DE RHAM cohomology groups $\mathbb{H}^{i}_{DR}(\mathcal{C}/T)$ of the family $\mathcal{C}/T$ are the hypercohomology groups of $C^{\cdot}(\Omega_{\mathcal{C}/T})$ defined as cohomology groups of the corresponding total Čech - de Rham complex $C^{tot}(\Omega_{\mathcal{C}/T})$. The construction applies to all restricted families $\mathcal{C}_U/U$, U an open part of T. On this way one gets the D E R H A M c o h o m o l o g y s h e a v e s $\mathcal{H}^{i}_{DR}(\mathcal{C}/T)$ on T.

We restrict ourselves now to curve families $\mathcal{C}/T$. For our purposes it suffices to assume that T is an affine part of a projective space $\mathbb{P}^N$. Let $\mathcal{D}_T$ be the sheaf of differential operators on T. Then the de Rham cohomology sheaf $\mathcal{H}^{1}_{DR}(\mathcal{C}/T)$ is not only an $\mathcal{O}_T$-module but also a $\mathcal{D}_T$-module sheaf. Looking for a family with a section $\bar{\omega}$ in $\mathcal{H}^{1}_{DR}(\mathcal{C}/T)$ satisfying the differential equations (0.1) with $\bar{\omega}$ instead of F one can take the P I C A R D c u r v e f a m i l y

$$\mathcal{C}/\mathbb{P}^2 \setminus \mathbb{A} \ : \ Y^3 = X(X-1)(X-u)(X-v)$$

and $\bar{\omega}$ represented by the differential form $\omega = dx/y$ depending on u,v. Taking integrals along cycles one gets an "algebraic" fundamental system of solutions

(4.1)

$$I_k(t) = \int_{\alpha_k(t)} \omega(t), \ k = 1,2,3, \ t = (u,v) \in \mathbb{P}^2 \setminus \mathbb{A}, \quad \omega = dx/y$$

Altogether we found the developing map of the FUCHSian system (0.1) in an explicit and algebraic manner. Looking back to the geometric

starting point and

to the results of the earlier sections we receive

4.2 Theorem. The quotient map $p: \mathbb{B} \longrightarrow \mathbb{P}^2$ with covering group $\Gamma(\sqrt{-3})$ is inverted by $(I_0 : I_1 : I_2) : \mathbb{P}^2 \setminus$ $\longrightarrow \mathbb{B}$ on $\mathbb{P}^2 \setminus$ with cycloelliptic integrals $I_k(t)$ described in (4.1) along linearly independent cycle families $\alpha_0(t)$, $\alpha_1(t)$, $\alpha_2(t)$.

4.3 Remark. Historically p^{-1} was first known [26]. The monodromy group was known to be a sublattice of $\Gamma(\sqrt{-3})$ generated by five elements, see [1].

5. Moduli Space of PICARD Curves.

5.1 Definition. A P I C A R D c u r v e is an algebraic (complex compact) curve isomorphic to one of the following plane curves of equation type

(5.2)

$$Y^3 = \prod_{i=1}^{4} (X - e_i) = X^4 + G_2 X^2 + G_3 X + G_4 \qquad \text{(affine)},$$

$$WY^3 = \prod_{i=1}^{4} (X - e_i W) = X^4 + G_2 W^2 X^2 + G_3 W^3 X + G_4 W^4 \qquad \text{(projective)},$$

Notice that $\sum_{i=1}^{4} e_i = 0$.

One proves that the n o r m a l f o r m (5.2) of a PICARD curve C is well-defined up to a common factor of the e_i's, if C is smooth, that means $e_i \neq e_j$ for $i \neq j$, see [14]. In other words:

5.3 Proposition-Definition. The m o d u l i s p a c e o f s m o o t h PICARD c u r v e s is $(\mathbb{P}^2 \setminus$ $)/S_4$. The m o d u l i s p a c e o f PICARD c u r v e s is $\mathbb{P}^2/S_4$.

6. The Relative SCHOTTKY Problem

Smooth PICARD curves C have genus 3. The Jacobian (variety) of C is denoted by J(C) and Jac(C) denotes the canonically polarized Jacobian of C. The correspondence $C \longmapsto \mathrm{Jac}(C)$ induces a birational map jac: $\mathbb{P}^2/S_4 \dashrightarrow \bar{\mathcal{A}}_3$, where $\bar{\mathcal{A}}_3$ is the moduli space of principally polarized abelian threefolds. Its restriction to $\mathbb{P}^2 \setminus$ is an open embedding by TORELLI's Theorem and 5.3. We want to uniformize the map jac in an effective manner, that means we look for a commutative diagram

(6.1)
$$\begin{tikzcd}
\mathbb{B} \arrow[rr, hook, "*"] \arrow[dd, "\Gamma"'] \arrow[dr, "th", "\Gamma(\sqrt{-3})"'] & & \mathbb{H}_3 \arrow[dd, "Sp(6{,}\mathbb{Z})"] \\
& \mathbb{P}^2 \arrow[dl, "S_4"'] \arrow[dr, dashed] & \\
\mathbb{P}^2/S_4 \arrow[rr, dashed, "\mathrm{jac}"'] & & \bar{\mathcal{A}}_3
\end{tikzcd}$$

The vertical arrows denote quotient maps by the arithmetic groups $\Gamma = U((2,1), \mathcal{O}_K)$ or $Sp(6,\mathbb{Z})$, respectively. The **relative SCHOTTKY problem for PICARD curves** asks for the explicit knowledge of $*$ in terms of period matrices.

The new ball $\mathbb{B} \subset \mathbb{P}^2$ corresponds to the hermitean (2,1)-metric on $\mathbb{C}^3$ defined by $\begin{pmatrix} 0 & 0 & \bar{\varrho} \\ 0 & 1 & 0 \\ \varrho & 0 & 0 \end{pmatrix}$; $\langle\ ,\ \rangle$ denotes the hermitean product.

Now we define successively the maps

(6.2)

$$*: \mathbb{C}^3 \longrightarrow \mathbb{C}^3, \ (A,B,C) \longmapsto (A,B,-\bar{\varrho}A,C,\bar{\varrho}\bar{B},\varrho C),$$

$$P: \mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^3 \longrightarrow \mathrm{Mat}_{3\times 6}(\mathbb{C}), \ (a,b,v) \longmapsto \begin{pmatrix} *a \\ \overline{*b} \\ \overline{*v} \end{pmatrix},$$

Π : the restriction of P to triples with the conditions:

$\langle a, a\rangle < 0, \ b \perp a, \ v \perp a, \ b, v$ linearly independent;

$$*: \mathbb{B} \longrightarrow \mathbb{H}_3, \ \mathbb{P}a \longmapsto Gl_3(\mathbb{C})\backslash \Pi(a, b, v)$$

The images of Π (or $*$) are called t y p i c a l p e r i o d m a t r i c e s (p o i n t s). The orthogonality conditions come from the RIEMANN period relations.

6.3 Theorem ([14]). A principally polarized abelian threefold is the Jacobian of a PICARD curve if and only if it corresponds to a typical period point in $\mathbb{H}_3$.

The proof needs the construction of t y p i c a l symplectic bases of $H_1(C,\mathbb{Z})$ and the use of GALOIS-invariant bases of $H^0(C,\Omega^1_C)$ on smooth PICARD curves C due to PICARD [26].

7. Effective TORELLI Theorem

We would like to make TORELLI's Theorem for PICARD curves effective by means of transcendent functions in analogy to the elliptic modular function j. Our problem is to find for given $\tau \in \mathbb{B}$ (or $*\tau \in *\mathbb{B} \subset \mathbb{H}_3$) the normal equation (see (5.2)) of a PICARD curve C_τ corresponding to the moduli point $p(\tau) \in \mathbb{P}^2$. In other words we look for an explicit analytic description of the quotient maps in diagram (6.1).

7.1 Definition. a holomorphic function $f: \mathbb{B} \longrightarrow \mathbb{C}$ is called a PICARD modular form (Γ' - form) of weight m, if there is a sublattice Γ' of a PICARD modular group such that

(7.2) $$\gamma^*(f) = j_\gamma^m \cdot f \text{ for all } \gamma \in \Gamma' ,$$

where $j_\gamma(\tau)$ is the JACOBI determinant of γ at $\tau \in \mathbb{B}$.

7.3 Theorem ([11]). There is a basis t_1, t_2, t_3 of the space of $S\Gamma(\sqrt{-3})$-forms of weight 1 such that

$$\tau \longmapsto (t_1(\tau) : t_2(\tau) : t_3(\tau) : t_4(\tau))$$

with $t_1 + t_2 + t_3 + t_4 = 0$ is the quotient map $p: \mathbb{B} \longrightarrow \mathbb{P}^2$.

The proof goes through the fine surface classification of $X = \mathbb{B}/S\Gamma(\sqrt{-3})$ and identification of the modular forms with ele- ~~sec-~~ ments ~~tions~~ of $H^0(\widetilde{X}, \mathcal{O}(mK_{\widetilde{X}} + mT)$, $\widetilde{X}$ the minimal smooth compactification and T the compactifying curve. The surface $\widetilde{X}$ is described by the weighted homogeneous equation

(7.4) $S^3 = (S_2^2 - S_1^2)(S_2^2 - S_0^2)(S_1^2 - S_0^2)$, $wt(S_i) = 1$, $wt(S) = 2$.

This is a nice example fitting in the 22-nd HILBERT PROBLEM again, see section 2. Namely, the ring of $S\Gamma(\sqrt{-3})$-forms is $\mathbb{C}[s_0, s_1, s_2, s]$ with generators satisfying the same relations as described in (7.4) The compactification theory of BAILY-BOREL [2] is the general framework of solution of HILBERT's 22-nd PROBLEM, by means of automorphic forms, but the explicit solution remains to be a case-by-case problem.

We change over from the left to the right quotient map in diagram (6.1) and guess that the theory of theta functions is helpful to describe the modular forms t_i in 7.3 more explicitly in terms of Fourier series as it is known for the elliptic modular function $j(\tau)$, see e.g. [21].

Theta functions $\vartheta\begin{bmatrix} a \\ b \end{bmatrix}$ with characteristics $a,b \in \mathbb{Q}^g$ are holomorphic functions on $\mathbb{C}^g \times \mathbb{H}_g$. Explicitly the theta functions

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix} : \mathbb{C}^g \times \mathbb{H}_g \longrightarrow \mathbb{C}$$

are defined by

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z,\Omega) = \sum_{n\in\mathbb{Z}^g} \exp\{\pi i\, {}^t(n+a)\Omega(n+a) + 2\pi i\, {}^t(n+a)(z+b)\}$$

The restrictions $\vartheta | 0 \times \mathbb{H}_g$

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(\Omega) = \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(0,\Omega)$$

are called *theta constants* (with characteristics).

7.5 Theorem (Feustel [6] , Shiga [39]).

Let $\theta_i(\Omega) = \vartheta_i(0,\Omega)$, $i = 0,1,2$, be the theta constants on $\mathbb{H}_3$ restricting the theta functions

$$\vartheta_k = \vartheta\begin{bmatrix} 0 & 1/6 & 0 \\ k/3 & 1/6 & k/3 \end{bmatrix}(z,\Omega)\ ,\ k = 0,1,2,\ z \in \mathbb{C}^3.$$

Set

(7.6) $$Th_1 = \theta_0^3 + \theta_1^3 + \theta_2^3\ ,\ Th = -3\theta_0^3 + \theta_1^3 + \theta_2^3\ ,$$

$$Th = \theta_0^3 - 3\theta_1^3 + \theta_2^3\ ,\ Th = \theta_0^3 + \theta_1^3 - 3\theta_2^3 .$$

~~and~~ Then

(7.7) $$th_i(\tau) = Th_i(*\tau),\ i = 1,2,3,4,\ \tau \in \mathbb{B},$$

is aspecial choice of $S\Gamma(\sqrt{-3})$-forms inverting PICARD's integral map $I: \mathbb{P}^2 \setminus \; \longrightarrow \mathbb{B}$ as ~~described~~ explained in 7.3.

7.8 Remark. The $S\Gamma(\sqrt{-3})$-forms th_i are n o r m a l i z e d to be compatible with the action of $\Gamma/S\Gamma(\sqrt{-3}) \cong S_4 \times (\mathbb{Z}/3\mathbb{Z})$, S_4 symmetric group, in the following sense (see [12] or [6]):

Up to a character the group $S_4 \times (\mathbb{Z}/3\mathbb{Z})$ acts on $\{th_1, \ldots, th_4\}$ via permutations of indices.

7.9 Corollary. Each PICARD curve C has a normal form

$$(7.10) \quad C_\tau:\; Y^3 = \prod_{i=1}^{4} (X - th_i(\tau)) = X^4 + G_2(\tau)X^2 + G_3(\tau)X + G_4(\tau)$$

for a suitable $\tau \in \mathbb{B}$.

FEUSTEL's proof of Theorem 7.7 needs preperatory work of RIEMANN (RIEMANN constants), PICARD [26], ALEZAIS [1], MUMFORD [24], SHIGA's preperatory work for [20] and HOLZAPFEL [11]. The idea is to show that some theta candidates coming out directly from PICARD curves satisfy all the functional equatiens defining Γ-forms of N e b e n t y p u s described in (7.2) and 7.8. Since generators of Γ are explicitly known, this is an effective finite problem.

8. Special Values of the PICARD Modular Theta Function

8.1 Definition. A s i n g u l a r m o d u l e on $\mathbb{B}$ is an isolated fixed point of (an element) of $U((2,1),K)$. If the Jacobian $J_\tau = J(C_\tau)$, $\tau \in \mathbb{B}$ can be decomposed up to isogeny into simple abelian varieties with complex multiplication, then τ is called a $\mathfrak{D}$CM - module (Decomposed Complex Multiplication).

8.2 Proposition (FEUSTEL, unpublished; maybe to read off from [9] with some effort). The point $\tau \in \mathbb{B}$ is singular module iff it is a $\mathfrak{D}$CM-module.

We denote the preimage of along $p: \mathbb{B} \longrightarrow \mathbb{P}^2$ by . It consists of infinitely many discs in $\mathbb{B}$ (see [11]).

8.3 Theorem

(A) ([14]). If $\sigma \in \mathbb{B}$ is a singular module, then

(8.4) $$\mathrm{th}(\sigma) = (\mathrm{th}_1(\sigma):\mathrm{th}_2(\sigma):\mathrm{th}_3(\sigma):\mathrm{th}_4(\sigma))$$

is an algebraic point of $\mathbb{P}^2$.

(T) ([30],[31]). If $\tau \in \mathbb{B}(\overline{\mathbb{Q}}) \setminus$, then $\mathrm{th}(\tau)$ is a transcendent point of $\mathbb{P}^2$, this means that $\mathrm{th}(\tau) \notin \mathbb{P}^2(\overline{\mathbb{Q}})$.

The first part is an application of the Theorem of SHIMURA-TANIYAMA stating the algebraicity of moduli points of (polarized) abelian CM-varieties. The second part comes out from WUESTHOLZ' transcendence theory, see [36], as announced in [14]. Conjecturally the exclusion of in (T) can be omitted by the method of jumping to elliptic curves used in [14] for the complete proof of (A).

Using more carefully the SHIMURA-TANIYAMA theory of complex multiplication of abelian varieties one discovers a strong quality of special values $\mathrm{th}(\tau)$ in the case (A) fitting in HILBERT's 12-th

PROBLEM.

8.5 Theorem (Explicit construction of SHIMURA class fields of cubic extensions of EISENSTEIN numbers by special values of Theta constants (see [18]). Let σ be a singular module such that ~~with simple~~ J_σ has complex multiplication. Then we have, with the notations below, a tower of algebraic number fields

$$F'_\sigma(th(\sigma)) \;/\; F'(th(\sigma))^{S_4(\sigma)} \;/\; F'_\sigma \;/\; K,$$

where the middle extension is abelian (SHIMURA class field), which is unramified, if the additional ideal condition (I) is satisfed.

8.6 Notations (see [32] or [22]). The endomorphism algebra $F_\sigma \cong \mathrm{End}(J_\sigma) \otimes \mathbb{Q}$ is the cubic extension $K(\sigma)$ of K. Its reflex field is is denoted by F'_σ. We set

$$F'_\sigma(th(\sigma)) = F'_\sigma(\ldots, th_i(\sigma)/th_j(\sigma), \ldots) \;,\; 1 \leqslant i,j \leqslant 4$$

in the parantheses. The symmetric group acts on the generators $th_i(\sigma)/th_j(\sigma)$ via permutation of indices. All such permutation, which are extendable to an automorphism of $F'_\sigma(th(\sigma))/F'$ form the the group $S_4(\sigma)$. Let (F_σ, Φ_σ), $\Phi_\sigma = \sum_{i=1}^{3} \varphi_i$ with field embeddings $\varphi_i : F \hookrightarrow \mathbb{C}$, be the type of $\mathrm{Jac}(C_\sigma)$. Then one has $J_\sigma \cong \mathbb{C}^3/\Phi_\sigma(\mathfrak{A}_\sigma)$ for a suitable $\mathbb{Z}$-lattice $\mathfrak{A}_\sigma$ of F_σ. The ideal condition in 8.5 is:

(I) $\mathfrak{A}_\sigma$ is a (fractional) ideal of F_σ.

9. Connection With FERMAT Equations

FREY [7] discovered a deep connection between FERMAT's Last

Theorem and the arithmetic of elliptic curves. PARSHIN [25] joint it with the BOGOMOLOV–MIYAOKA–YAU inequality transfered to arithmetic surfaces. The corresponding inequality for invariants of arithmetic surfaces is not proved until now. We will call it the PARSHIN problem (PAR), see below. PARSHIN proved the following remarkable

9.1 Theorem ([9]). If the PARSHIN problem has an affirmative answer, then the statement of FERMAT's Last Theorem is true for almost all FERMAT equations $X^p + Y^p = Z^p$.

"Almost all" means: all up to a finite number. The conclusing statement of 9.1 is also known as "Asymptotic FERMAT Theorem". Since the proof goes through elliptic curves, we asked for a similar connection with Picard curves. Surprisingly we found a rigorous reduction of PARSHIN's Theorem 9.1, which could be useful for a more effective approach. We announce

9.2 Theorem ([19]). If the PARSHIN problem has an affirmative answer only for arithmetic surfaces of KODAIRA–PICARD type, then the asymptotic FERMAT Theorem holds.

We finish with the presentation of some necessary definitions and hints for the proof. The best reference is LANG's book [23].

Let X/S be an arithmetic surface. $B = \operatorname{Spec} \mathcal{O}$, $\mathcal{O} = \mathcal{O}_L$; L a number field. We say that X/B is of KODAIRA - PICARD type, if the general fibre X_L is a PICARD curve or a smooth biquadratic covering of a smooth PICARD curve branched over exactly one point. PARSHIN's problem is to prove

(PAR) There are universal constants a_0, a_1, a_2 (a_0 depending on the genus g of general fibre) such that for all semistable arithmetic surfaces X/B it holds that

(Ar.BMY) $$(\omega^2_{X/B}) \leq a_2 \sum_{v \in M_L} \delta_v + a_1 (2g-2) \log |D_{L/\mathbb{Q}}| + a_0.$$

$D_{L/\mathbb{Q}}$ the discriminant

$(\omega^2_{X/B})$ is the selfintersection of the relative canonical sheaf, M_L is the set of all (finite and infinite) places of L. For δ_v we refer to [25]. The most complicated contributions δ_v, $v \in M_\infty$) are described in terms of special values of Theta constants in the framework of FALTINGS' basic theory of arithmetic surfaces [5].

For the proof of 9.2 we followed the line of VOJTA's reduction in [37] of PARSHIN's Theorem 9.1 to KODAIRA-PARSHIN covers of the FERMAT curve of genus 3.

Until now an explicit calculation of (global and local) invariants of arithmetic surfaces needed in (Ar.BMY) seems to be only possible for the elliptic case ([5]) and genus 2 ([4]) because of a good knowledge of the connections with modular forms. A rush to the KODAIRA-PICARD types would accomplish one side of the deep mathematical thinking and feeling of the old masters PICARD and HILBERT.

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